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Unified geometric and analytical treatment of magnetogasdynamic shocks. Part 1. General solutions and theorems

By LEE A. BERTRAM

Department of Engineering Mechanics and Engineering Research Institute,
Iowa State University, Ames

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In this paper, the first of two parts on solutions to the non-relativistic de Hoffmann–Teller relations, a unified notation is developed and used to produce new analytical solutions for one-dimensional unsteady and two-dimensional crossed-fields representations. A simple numerical solution method for the non-aligned two-dimensional representation is also included. These solutions are explicit in the upstream parameters and a single strength parameter. Unification of reality, entropy and evolutionary conditions is also carried out. The resulting single criterion is used to derive algorithms by which the correct solution is obtained, in every case without previous computation and elimination of non-physical solutions. Classifications of the solutions are discussed in relation to previous schemes.

1. Introduction

This paper examines the problem of obtaining solutions for the state downstream from a prescribed magnetogasdynamic shock. In this approach, the geometric properties of the shock are exploited to connect the one-dimensional unsteady, the non-aligned two-dimensional steady and the aligned-fields representations, without resorting to formal Galilean transformation (§2). This approach not only allows discussion of all cases from a single point of view, but suggests adoption of a unified notation, in which those variables which are Galilean invariants are used to write solutions in all representations.

A solution here will refer to an explicit analytical function giving the downstream state in terms of the fixed parameters of the upstream state and *a single strength parameter*. Such a form of solution allows exhaustive classification of the possible types of shock, thereby allowing a possibility for overall comprehension of shock behaviour. It also provides a basis for the efficient organization of the inevitable numerical evaluations of the solutions in practical use. The emphasis on the use of a single strength parameter follows from the difficulty in realizing the above-listed benefits if two or more are used (see discussion in Bertram 1973). In fact, even a numerical study formulated so that only a single strength parameter appears, as in the non-aligned two-dimensional studies of Lynn (1966) and of Morioka & Spreiter (1969), can give much information by indirect means and may prove more useful than an analytical formulation with multiple strength parameters.

The only solutions of a general nature conforming to the above definition are those of Bazer & Ericson (1959) for the one-dimensional unsteady representation and the author's (Bertram 1973) for the aligned-fields case. Further examples are derived in § 3, where an explicit solution is given for the crossed-fields case. Also given is a new single-valued form of the one-dimensional unsteady solution, with the downstream value of the ratio of normal relative velocity to the normal Alfvén wave speed appearing as the strength parameter. Besides being convenient, this latter solution suggests and allows proof of a pair of theorems on the relation of reality, entropy and evolutionary conditions in § 4. These three criteria are then unified, in the sense that a single criterion equivalent to all three is presented. In the interest of brevity, detailed proofs have been omitted, except for these two central theorems.

For the two-dimensional steady representation of the shock, no analytical solution was found, except in the aligned- and crossed-fields limits. However, because of the simplicity of the unified reality–entropy–evolutionary condition, even this case can be included in the discussion in § 5, where explicit bounds are given on the strength parameter in all representations. When a value of the strength parameter is chosen within the given bounds, the resulting solution is guaranteed to be real-valued, compressive and evolutionary. The only complication for the numerically solved two-dimensional case is that a specific algorithm must also be given to obtain the correct root from the polynomial connecting the downstream state and strength parameter. These bounds, like those of Bazer & Ericson (1959) and Ericson & Bazer (1960), are written in terms of the upstream parameters only, and do not contain the strength parameter.

Finally, in § 6, the problem of classification of solutions, considered by all earlier authors, notably de Hoffmann & Teller (1950), Friedrichs (1954), Friedrichs & Kranzer (1958) and Bazer & Ericson (1959), is reconsidered from the present point of view. The resulting system is essentially that of Bazer & Ericson (1959), but is restricted to the classification of individual shock properties, with shock polar classification deferred to part 2 of this study. Hopefully, this simpler system enhances the clarity of the earlier schemes.

2. Nomenclature and geometry

Consider a region filled with a perfectly conducting, electrically neutral, inviscid, perfect gas with gas constant R, constant specific heat ratio γ, such that $1 < \gamma < 2$, and constant magnetic permeability μ. The gas has mass density ρ_0^+, absolute pressure and temperature p_0^+ and T_0^+, and specific entropy S_0^+. It moves at a non-relativistic velocity $\mathbf{u}_0^+$ through a magnetic field with induction vector $\mathbf{B}_0^+$. Let a plane shock surface, moving at non-relativistic velocity $\mathbf{U}^+$, penetrate the region. Select Cartesian co-ordinate directions with unit vector $\mathbf{n}$ normal to the shock pointing downstream in the $-\mathbf{U}^+$ direction, unit vector $\mathbf{k}$ in the z direction, upwards out of the plane containing $\mathbf{U}^+$ and $\mathbf{B}_0^+$, and unit vector $\mathbf{t} = \mathbf{k} \times \mathbf{n}$ tangential to the shock surface (figure 1). The superscript + indicates

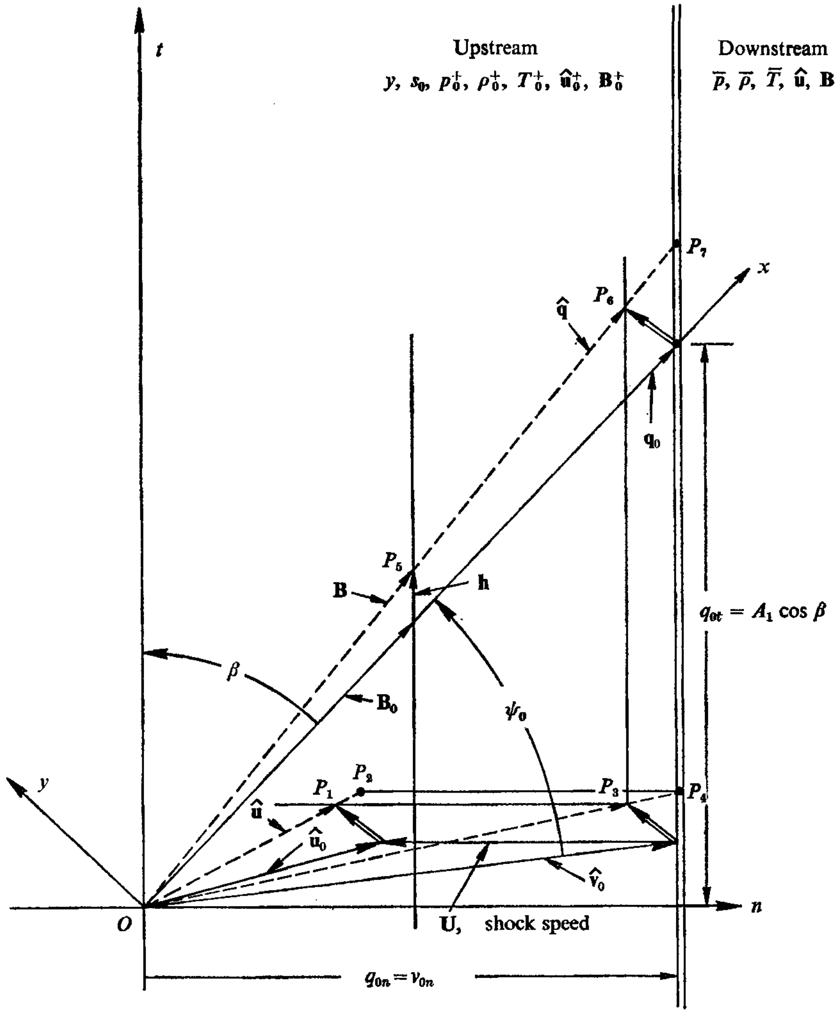


FIGURE 1. Shock geometry in the plane of $\mathbf{U}$ and $\mathbf{B}_0$. $\rightarrow$, upstream vector; $\dashrightarrow$, downstream vector; $\Rightarrow$, $[\![\mathbf{u}]\!] = \mathbf{u} - \mathbf{u}_0$; $=$, shock surface.

a dimensional variable in rationalized mks units, while the subscript zero or no subscript denotes upstream and downstream values, respectively, of the variable.

The discontinuities of the variables through the shock must satisfy (de Hoffmann & Teller 1950)

$$[\![\rho^+ v_n^+]\!] = 0, \tag{2.1}$$

$$[\![\rho^+ v_n^+ u_n^+ + p^+ + (B_t^{+2} - B_n^{+2})/2\mu]\!] = 0, \tag{2.2}$$

$$[\![\rho^+ v_n^+ u_t^+ - B_n^+ B_t^+/\mu]\!] = 0, \tag{2.3}$$

$$[\![u_z^+]\!] = 0, \tag{2.3a}$$

$$\left[\!\!\left[\rho^+ v_n^+ \left(\tfrac{1}{2}\mu^{+2} + \frac{p^+/\rho^+}{\gamma - 1} + \frac{B^{+2}}{2\mu\rho^+}\right) + u_n^+ \left(p^+ + \frac{B^{+2}}{2\mu}\right) - \frac{(\mathbf{u}^+ . \mathbf{B}^+) B_n^+}{\mu}\right]\!\!\right] = 0, \tag{2.4}$$

$$p^+ = \rho^+ R T^+ = p_0^+ (p^+/\rho_0^+)^\gamma \exp\left[(\gamma - 1)(S^+ - S_0^+)/R\right], \tag{2.5}$$

$$[\![B_n^+]\!] = 0, \tag{2.6}$$

$$[\![\mathbf{E}^{+\prime} - E_n^{+\prime}\,\mathbf{n}]\!] = [\![\mathbf{B}^+ \times \mathbf{v}^+ - u_z^+ B_t^+ \mathbf{n}]\!] = 0, \tag{2.7}$$

$$[\![E_n^{+\prime}]\!] = [\![\mathbf{B}_t^+ \times u_z^+ \mathbf{k}]\!] = e_s^{+\prime}/\epsilon \tag{2.8}$$

and

$$[\![\mathbf{n} \times \mathbf{B}^+]\!] = \mu j_s^+, \tag{2.9}$$

where $[\![X]\!] = X - X_0$ and $\mathbf{v} = \mathbf{u}^+ - \mathbf{U}^+$ is the gas velocity relative to the shock; $\mathbf{j}_s^+$ is the current density per unit length of shock. The electrostatic charge density per unit area of the shock $e_s^{+\prime}$ appears divided by the constant permittivity of the gas ϵ, so $e_s^{+\prime}/\epsilon$ is of the order of the electric field vector $\mathbf{E}^{+\prime}$ discontinuity; the prime indicates that these variables are seen from co-ordinates moving with the shock, as is $\mathbf{v}$.

Dimensionless variables are introduced by

$$\overline{p} = p^+/p_0^+, \quad \overline{\rho} = \rho^+/\rho_0^+, \quad \overline{T} = T^+/T_0^+, \quad [\![S]\!] = (S^+ - S_0^+)/R,$$

$$\mathbf{B} = \mathbf{B}^+/B_0^+ \quad \text{and} \quad s_0 = a_0^{+2}/b_0^{+2},$$

where $b_0^+ = B_0^+/(\mu\rho_0^+)^{\frac{1}{2}}$ is the upstream Alfvén speed and $a_0^{+2} = (\gamma p_0^+/\rho_0^+)^{\frac{1}{2}}$ is the upstream acoustic speed. All velocity vectors are made dimensionless by division by b_0^+. Since $\mathbf{B}$ can undergo only tangential jumps according to (2.6), it is also useful to introduce $\mathbf{h} = [\![\mathbf{B}]\!] = (\mathbf{B}_t^+ - \mathbf{B}_{0t}^+)/B_0^+$. In terms of these variables, the problem is to solve (2.1)–(2.9) for $\mathbf{h}$, $[\![\mathbf{u}]\!]$, $\overline{\rho}$, $\overline{p}$, $\overline{T}$ and $[\![S]\!]$ as functions of the upstream variables γ, s_0, $\mathbf{B}_0$, $\mathbf{u}_0$ and some strength parameter, $\mathbf{U}$ or its equivalent. Before such a solution is written down, some general observations are made.

Because the de Hoffmann–Teller shock equations (2.1)–(2.9) are non-relativistic, they are, by definition, Galilean invariant. In particular, an arbitrary Galilean transformation leaves the scalars γ, s_0, $\overline{p}$, $\overline{\rho}$, $\overline{T}$ and $[\![S]\!]$, the vectors $\mathbf{B}$, $\mathbf{h}$, $\mathbf{j}_s$ and $[\![\mathbf{u}]\!]$, and the geometric quantities v_{0n} and β unchanged; the angle β is measured positive counterclockwise from $\mathbf{B}_0$ to $\mathbf{t}$. Both the velocity $\mathbf{u}$ of the one-dimensional unsteady (henceforth one-dimensional) representation and velocity $\mathbf{v}$, with its associated angle ψ_0 measured counterclockwise from $\mathbf{v}_0$ to $\mathbf{B}_0$ of the non-aligned two-dimensional steady (henceforth two-dimensional) representation, change both in magnitude and inclination. Thus, it is to be expected that a solution written in terms of the invariants listed above will prove useful, regardless of the co-ordinate system from which the shock is viewed, while a solution explicit in velocities cannot.

The variables $\mathbf{E}' = \mathbf{E}^{+\prime}/b_0^+ B_0^+$, $\mathbf{j}_s = \mu \mathbf{j}_s^+/B_0^+$, $e_s' = e_s^{+\prime}/\epsilon b_0^+ B_0^+$ and u_z are all secondary variables, in the sense that they do not appear in (2.1)–(2.7) after u_z has been dropped from (2.4) and (2.7) by use of (2.1) and (2.3a). Thus (2.1)–(2.7) suffice to determine the seven unknowns u_n, u_t, B_n, B_t, $\overline{\rho}$, $\overline{p}$ and $\overline{T}$ in terms of the upstream parameters and the strength parameter, and further, these seven unknowns satisfy the same equations regardless of the value of u_z. Therefore, all Galilean transformations in the z direction are seen to be trivial, in that they modify only the values of the secondary variables e_s', E_n', E_t' and u_z. This conclusion allows us to discuss the solution considering only velocity components in the n, t plane, that is, to replace $\mathbf{u}$ and $\mathbf{v}$ by $\hat{\mathbf{u}} = \mathbf{u} - u_z\mathbf{k}$ and $\hat{\mathbf{v}} = \mathbf{v} - v_z\mathbf{k} = \mathbf{v} - u_z\mathbf{k}$, etc., as is well known. It also allows us to generalize slightly the aligned-fields label, since it is necessary only that the in-plane velocity and induction vectors are parallel upstream for the full simplification associated with the aligned-fields representation to be exploited as is done below.

Now, of the infinite number of one- and two-dimensional representations available for the same shock, there are two which are physically distinguished from all others by the condition $E_{0z} = 0$; the 'one-dimensional' co-ordinates which move with the gas in the plane so that $v_{0n} = U$ and $\mathbf{u}_0 = 0$, and the 'two-dimensional' aligned-fields co-ordinates in which the upstream gas velocity is $\hat{q}_0 = v_{0n}(\mathbf{n} + (\cot\beta)\,\mathbf{t}) = (v_{0n}/\sin\beta)\;\mathbf{i}$, the co-ordinates (x, y) being, respectively, parallel and normal to $\mathbf{B}_0$ in the plane.

These two representations also share the property that (2.1)–(2.7) contain only invariant terms in these co-ordinates. Therefore, it is to be expected that solutions written in terms of these variables will play a central role in any discussion of the shock. This invariance is clear for the one-dimensional system with $\mathbf{u}_0 = \mathbf{0}$ since $[\![\mathbf{u}]\!] = \mathbf{u}$ is true in that case. For the aligned-fields representation, note that the solution will be written in terms of γ, s_0, A_1, $\bar{\rho}$ and β, all of which have already been shown to be invariant except possibly A_1, the component of $\mathbf{q}_0$ along $\mathbf{B}_0$. But, because $q_{0n} = v_{0n}$ defines $\hat{\mathbf{q}}_0$, we have

$$A_1 = v_{0n}/\sin\beta = \pm\,|\hat{\mathbf{q}}_0|, \tag{2.10}$$

which must be invariant since it is the ratio of invariants. The notation A_1 has been used because $A_1^2 = \hat{q}_0^{+2}/b_0^{+2}$, the upstream Alfvén number squared of the aligned-fields representation, and $A_1^2 = (v_{0n}^+/b_{0n}^+)^2$, the one-dimensional Alfvén number based on normal speeds. The subscript 1 is used to emphasize that A_1 is a strength parameter equivalent to U or v_{0n} for the one- or two-dimensional formulations. The sign of A_1 is positive or negative, respectively, for $\hat{\mathbf{q}}_0$ parallel or anti-parallel to $\mathbf{B}_0$; this means that A_1 is not simply the Alfvén number.

Part of the role played by the aligned-fields representation may be seen by combining (2.1), (2.6), (2.7) and (2.10) to obtain

$$\mathbf{q}_0 = A_1\mathbf{B}_0, \quad \bar{\rho}\mathbf{q} = A_1\mathbf{B}; \tag{2.11}$$

that is, the alignment of mass flux and induction vectors survives the shock with the same constant of proportionality A_1 (de Hoffmann & Teller 1950). This relation can be given the following simple and useful geometric interpretation. For a prescribed upstream state and strength parameter, $[\![\mathbf{u}]\!]$ is determined from (2.1)–(2.7). Owing to its Galilean invariance, $[\![\mathbf{u}]\!] = [\![\mathbf{v}]\!] = [\![\mathbf{q}]\!]$, and thus P_6 of figure 1 is known when $[\![\mathbf{u}]\!]$ is known. Because of (2.11), $\mathbf{B}$ lies along OP_6, and (2.6) requires its tip to lie on the line through the tip of $\mathbf{B}_0$, parallel to $\mathbf{t}$. The intersection of these two lines at P_5 must then be the tip of $\mathbf{B}$ downstream of the shock. Also, figure 1 displays the geometric interpretation of (2.1). Extension of the vectors $\mathbf{v}$ and $\mathbf{q}$ until they intersect the shock surface at P_4 and P_7 produces the downstream mass flux vectors. Because of the dimensionless variables used here, upstream mass flux and velocity vectors are identical. The analytical form of this relation follows from (2.11) as

$$\mathbf{h} = [\![\rho\mathbf{q}]\!]/A_1. \tag{2.11a}$$

Thus, $\mathbf{h}$, $[\![\rho\mathbf{q}]\!]$ and $[\![\mathbf{u}]\!]$ are all related geometrically and analytically via the aligned-fields representation.

In summary, figure 1 displays the geometry of a single shock viewed from three different co-ordinate systems: first, an arbitrary one-dimensional system in which the gas has upstream velocity $\mathbf{u}_0$; second, an equally arbitrary two-dimensional system with upstream gas velocity $\mathbf{v}_0$; and finally, the unique and invariant aligned-fields co-ordinates. These representations all share values of the above-listed invariants, and are geometrically related through (2.10) and (2.11). This unity of notation and geometry allows the construction of shock solutions for all three cases from a single viewpoint, and connects them without inconvenient formal Galilean transformations.

3. Solutions

The precise task undertaken in this paper is to write solutions in a convenient form which enables us to construct the locus of all possible downstream states which can be reached from a fixed upstream state. Fixed means that all the upstream parameters as seen from the arbitrarily chosen co-ordinate system are held constant while a *single* strength parameter is varied. For any system, γ, s_0 and $\mathbf{B}_0 = \mathbf{i}$ are all held constant; in addition, for an arbitrary one-dimensional system β and $\mathbf{u}_0$ are constant while a strength parameter equivalent to v_{0n} varies; for arbitrary two-dimensional co-ordinates, ψ_0 and $A_0 = |\mathbf{v}_0|$ are constant while a strength parameter equivalent to β varies; and finally, for the aligned-fields case, A_1 and $\psi_0 = 0$ are constant and the strength parameter varies.

For the aligned-fields case, this task has been completed (Bertram 1973) by employing $\bar{\rho}$, another invariant, as a strength parameter. The closed-form solution is

$$\cos^2\beta = -\frac{\gamma+1}{A_1^2 M_1^{*2}}\frac{(\bar{\rho}-M_1^{*2})(\bar{\rho}-A_1^2)^2}{D(\bar{\rho},A_1^2,\gamma)}, \tag{3.1}$$

where $$D(\bar{\rho},A_1^2,\gamma) = \gamma\bar{\rho}^2-[\gamma+2+(\gamma-1)A_1^2]\bar{\rho}+(\gamma+1)A_1^2$$

and $$\bar{B}^2-1 = \frac{\gamma+1}{M_1^{*2}}\frac{(\bar{\rho}-1)(\bar{\rho}-M_1^{*2})[(2-A_1^2)\bar{\rho}-A_1^2]}{D(\bar{\rho},A_1^2,\gamma)}, \tag{3.1a}$$

where $M_1^{*2} = q_0^{+2}/a_0^{*2} = (\gamma+1)A_1^2/[2s_0+(\gamma-1)A_1^2]$ is the square of the Mach number based on the critical acoustic velocity a_0^* of the upstream flow and $M_1^{*2} < (\gamma+1)/(\gamma-1)$ is always true. Since the remaining downstream variables are easily determined in this or any other representation once the first two have been written down, we need only consider these two relations in their various forms.

Now, for the one-dimensional solution, we wish to treat β as a known constant and solve for $\bar{\rho}$ or A_1^2 in terms of the other. Equivalently, we may introduce the downstream Alfvén number squared

$$\xi = A_1^2/\bar{\rho} = v_n^2/b_n^2 = v_{0n}^{+2}/(b_{0n}^{+2}\bar{\rho}) \tag{3.2}$$

as a strength parameter and

$$\eta = \bar{\rho}-1 \tag{3.2a}$$

as an unknown, and then solve (3.1) to obtain

$$\eta = \frac{2}{\gamma-1}\frac{\xi-1}{\xi}\frac{\xi^2\sin^2\beta-(1+s_0)\,\xi+s_0}{(\xi-1)^2-\xi[\xi-\gamma/(\gamma-1)]\cos^2\beta}. \tag{3.3}$$

From (2.3) and (2.11) it is found that

$$[\![\rho\mathbf{q}]\!] = A_1\mathbf{h} = \bar{\rho}[\![\mathbf{q}_t]\!]+\eta\mathbf{q}_{0t} = \bar{\rho}\mathbf{h}/A_1+\eta\mathbf{q}_{0t} + A_1\mathbf{h}/\xi+\eta\mathbf{q}_{0t},$$

where $\mathbf{q}_{0t} = A_1\cos\beta\mathbf{t}$, so that

$$[\![\mathbf{B}]\!] = \mathbf{h} = \frac{\eta\xi\cos\beta}{\xi-1}\mathbf{t} = \frac{2}{\gamma-1}\frac{\xi^2\sin^2\beta-(1-s_0)\,\xi+s_0}{(\xi-1)^2-\xi[\xi-\gamma/(\gamma-1)]\cos^2\beta}\mathbf{t}, \tag{3.3a}$$

which, together with (3.3), is the desired one-dimensional solution. It is equivalent to the earlier solution of Bazer & Ericson (1959), but proves much more convenient to use. A comment on notation is also in order; $h = \mathbf{h}\,.\,\mathbf{t}$ here corresponds to Bazer & Ericson's h_f or their $-h_s$, while β here is the same as β for a fast shock or $\beta-\pi$ for a slow shock in the earlier aligned-fields study (Bertram 1973). This use of β and h, and also of the signed A_1 in (2.10) is necessary to obtain simple forms of solution such as (3.3), so that they apply to both fast and slow shocks. These latter are distinguished from one another simply by $h > 0$ and $h < 0$, respectively, for compressive shocks, according to (3.3a) and the definitions of fast and slow shocks in §4.

When an arbitrary two-dimensional co-ordinate system is used, two quantities which are not Galilean invariants must be held constant, namely the magnitude A_0 and inclination ψ_0 of $\hat{\mathbf{v}}_0$. Because the aligned-fields and two-dimensional representations have the same normal component of velocity

$$A_1 = A_0\sin(\beta+\psi_0)/\sin\beta. \tag{3.4}$$

Thus, specification of β or A_1 as a strength parameter determines both values in (3.1), leaving the task of solving a cubic for $\bar{\rho}$ (or a sixth-degree polynomial for A_1 or $\tan\beta$; see Lynn's (1966) equation (8a); his $\beta = \beta+\psi_0$ in our notation). This difficulty has appeared for every choice of dependent variable and parameter examined. The final relation is always cubic in the unknown, preventing a closed-form solution for this case.

It can be noted that this same result has been obtained in all previous non-aligned two-dimensional studies. The Germain–Shercliff (Germain 1959; Shercliff 1960) generalization of the Rayleigh and Fanno processes, as extended by Morioka & Spreiter (1969), yields three possible downstream states for each specified upstream state. Lynn's (1966) equations for velocity turning angle or $\bar{\rho}$, with coefficient functions of β, are cubic; so is the equation for the combination-of-angles parameters employed by Morioka & Spreiter. It might be conjectured that the basic difficulty in all these formulations is the use of A_0 and ψ_0, which lack Galilean invariance, as constant parameters (they are implicit in the Rayleigh process analysis, of course). Further, since this cannot be avoided, some cubic will always have to be solved. Despite this difficulty, a great deal can be learned by indirect analytical means, and particular solutions can be obtained, as in the past, by solving (3.1) numerically; this is the approach taken in part 2 of this study.

Despite the cubic governing equation for the general non-aligned two-dimensional case, there are two special values of ψ_0, namely $\psi_0 = 0$ (aligned fields) and $\psi_0 = \frac{1}{2}\pi$ (crossed fields), for which explicit solutions can be obtained. For crossed fields, (3.4) becomes $A_1 = A_0 \cot\beta$, and β may be eliminated from (3.1) to give

$$A_1^2 = A_0^2 \cot^2\beta = -\frac{(\gamma+1)(\bar{\rho}-M_1^{*2})(\bar{\rho}-A_1^2)^2}{\bar{\rho}P_\pi(\gamma, s_0, A_1^2, \bar{\rho})} A_0^2,$$

where

$$\begin{aligned} P_\pi &= [A_1^2 M_1^{*2} D(\gamma, A_1^2, \bar{\rho}) + (\gamma+1)(\bar{\rho}-M_1^{*2})(\bar{\rho}-A_1^2)^2]/\bar{\rho} \\ &= (\gamma+1)\bar{\rho}^2 + [\gamma A_1^2 M_1^{*2} - (\gamma+1)(M_1^{*2}+2A_1^2)]\bar{\rho} + (\gamma+1)A_1^2(A_1^2+2M_1^{*2}) \\ &\quad -[\gamma+2+(\gamma-1)A_1^2]A_1^2 M_1^{*2}. \end{aligned}$$

Finally, eliminating $\bar{\rho}$ by the substitution $\bar{\rho} = A_1^2/\xi$ gives

$$f_1 A_1^4 + f_2 A_1^2 + f_3 = 0, \tag{3.5}$$

where $f_i = f_i(\gamma, s_0, A_0^2, \xi)$ is as follows:

$$\begin{aligned} f_1 &= \gamma - 1 + (2-\gamma)\xi, \\ f_2 &= [2s_0 + (\gamma-1)A_0^2](\xi-1)^2 + \gamma\xi^2 - (\gamma+1)\xi, \\ f_3 &= (\xi-1)^2[2s_0 - (\gamma+1)\xi]. \end{aligned}$$

Because $A_1^2 = \bar{\rho}\xi = A_0^2\cot^2\beta$, equation (3.5) can be viewed as a quadratic for any of these variables. Since all are inherently non-negative, and since both roots of (3.5) can be shown to be negative if f_3 is positive,

$$\xi \geqslant 2s_0/(\gamma+1) \tag{3.5a}$$

is always required. Further, when (3.5*a*) is satisfied, the roots of (3.5) are of opposite sign, since f_1 is positive for $1 < \gamma < 2$. Thus, only the root

$$A_1^2 = \bar{\rho}\xi = A_0^2\cot^2\beta = [-f_2 + (f_2^2 - 4f_1f_3)^{\frac{1}{2}}]/2f_1 \tag{3.5b}$$

can be of physical interest. Therefore, A_1^2, $\cot^2\beta$ and $\bar{\rho}$ are all single-valued functions of ξ as given by (3.5*a*) and (3.5*b*), and all other variables can be readily written down.

Since both the aligned-fields solution (3.1) and the one-dimensional solution (3.3) derived from it are singular for particular cases, solutions in other variables must be offered for those cases. When the denominator function D in (3.1) vanishes, alternative solutions using β as a strength parameter have been presented in Bertram (1973). However, when (3.1) is written in the form $P(\eta) = 0$ in §5, this equation is shown to possess three distinct real roots, even when $D = 0$. This indicates that the numerical solution for the two-dimensional case is non-singular.

From figure 1, it is readily seen that there is one case for which both two-dimensional non-aligned and one-dimensional representations exist, but the aligned-fields representation does not, namely $v_{0n} \neq 0$ while $\beta = 0$. This is de Hoffmann & Teller's (1950) perpendicular shock, for which (see Bazer & Ericson 1959, equations 71)

$$h = \eta \tag{3.6}$$

and

$$(2-\gamma)\bar{\rho}^2 + [2s_0 + \gamma + (\gamma-1)v_{0n}^2]\bar{\rho} - (\gamma+1)v_{0n}^2 = 0. \tag{3.6a}$$

This solution may also be obtained as the large-ξ limit of (3.3a), which gives (3.6), and, once $h = \eta$ is substituted into (2.1)–(2.7), equation (3.6a) follows. Because (3.6a) is quadratic and its first and third coefficients are of opposite signs for $\gamma < 2$, there are always two real roots of opposite sign. The negative root is excluded as non-physical, so (3.6a) defines a unique positive $\bar{\rho}$, for which $\bar{\rho} \geqslant 1$ when $v_{0n}^2 \geqslant 1+s_0 = c_{f0}^2$.

With the inclusion of (3.6), the two-dimensional non-aligned solution from (3.1) and (3.4) is complete and non-singular, as is the crossed-fields solution (3.5). However, the one-dimensional solution requires a second addition because (3.3) is singular for $\xi = \sin^2\beta = 1$; that is (3.3) is singular for the normal shock with Alfvénic downstream flow – the switch-on shock (Friedrichs & Kranzer 1958; Bazer & Ericson 1959). In the present notation, with A_1^2 as strength parameter, the solution is

$$\bar{\rho} = A_1^2 \tag{3.7}$$

and

$$h^2 = (\gamma+1)(A_1^2-1)(1-A_1^2/A_1^{*2}), \tag{3.7a}$$

which has real h if and only if

$$1 \leqslant A_1^2 \leqslant A_1^{*2} \equiv (\gamma+1-2s_0)/(\gamma-1) \quad (s_0 \geqslant 1),$$

or

$$A_1^{*2} \leqslant A_1^2 \leqslant 1 \quad (s_0 \leqslant 1); \tag{3.7b}$$

obviously, $\bar{\rho} \geqslant 1$ only for $s_0 \leqslant 1$. A_1^{*2} is the value of A_1^2 for which $A_1^2 = M_1^{*2}$, that is, when $b_{0n}^{+2} = a_0^{*2}$. Finally, then, the complete one-dimensional solution consists of (3.3) with (3.6) and (3.7) appended.

4. Reality, entropy and evolutionary conditions

Before examining these criteria, it is convenient to introduce the classifications fast and slow for shock solutions, according to the definition of Bazer & Ericson (1959). A shock is called fast (or slow) if it is super-Alfvénic (sub-Alfvénic) downstream; that is, when $\xi > 1$ ($\xi < 1$). This definition allows clarity in the discussion of the relations between the conditions, and has the conventional meaning when applied to compressive and evolutionary solutions. The difficulty with the definitions employed, for example, by Jeffrey & Taniuti (1964) or Kulikovskiĭ & Lyubimov (1962), is that they equate these terms with the evolutionary condition. Friedrichs & Kranzer's (1958) original definition is not convenient because it involves terms from both sides of the shock.

Now, in order to exclude non-physical branches of the formal solutions given in § 3, the following criteria are applied. Since they are all written in terms of invariants, the results hold without change for all co-ordinate systems connected to the aligned system by Galilean transformation.

(a) *Reality.* Since each representation of the shock specified some set of values upstream and some strength parameter, these variables are automatically real-valued. However, they are chosen differently for each representation, and different authors have formulated the same representation differently; true generality in identification of 'automatically real' variables is very difficult. In addition, it would be of questionable utility to discuss the very broad classes of formal solutions for which some variables are complex-valued, since these can be eliminated very easily by direct inspection of the governing equations; e.g. complex β in the crossed-fields case is excluded by (3.5a).

Accordingly, *the approach taken here is not completely general*, but consists of breaking down the reality condition into two parts. The first part is to require that the problem be formulated so that *γ, s_0, $\mathbf{B}_0$ and β are real from the outset*; *in addition, either $A_1^2 \geqslant 0$ or $\xi \geqslant 0$ is satisfied from the outset.* This list contains those variables most frequently in the 'automatically real' category for the formulations in the literature, including the one-dimensional and non-aligned two-dimensional formulations above. For crossed-fields, real β imposes restriction (3.5a) on the strength parameter ξ, and for the aligned-fields case, real β requires some rather complicated restrictions on $\bar{\rho}$ (Bertram 1973). Therefore, the following discussion applies to these cases only when the strength parameters have values in these restricted regions. When these two conditions have been satisfied, the solution will be considered properly formulated.

After proper formulation of the solution, it is necessary only to write down all the remaining variables explicitly (see part 2) to see that the second part of the reality condition may be stated in the following two-level form.

(a, i) All variables, except possibly entropy, are real-valued if and only if

$$0 \leqslant \bar{\rho} < (\gamma+1)/(\gamma-1).$$

(a, ii) All variables, including entropy, are real if and only if $\bar{p} > 0$.

(b) *Non-decreasing entropy.* Lüst's (1955; see Ericson & Bazer (1960) for detailed proof) form $\bar{\rho} \geqslant 1$ is used here to guarantee $[\![S]\!] \geqslant 0$.

(c) *Evolutionary condition.* For shock stability, the evolutionary criterion (Akhiezer, Lyubarskiĭ & Polovin 1958; Polovin 1961), (c, i) $v_{0n} \geqslant c_{f0}$ for fast ($\xi > 1$) shocks or $b_{0n} > v_{0n} \geqslant c_{s0}$ for slow ($\xi < 1$) shocks upstream and (c, ii) $v_n \leqslant c_f$ for fast shocks or $v_n \leqslant c_s$ for slow shocks downstream will be applied as a necessary condition. Apparently it is still not known whether this is a sufficient condition. The fast and slow characteristic speeds, c_f and c_s respectively, are defined by (4.2a) below.

When a particular solution satisfies all these criteria, it will be labelled proper; if any criterion is violated the solution will be labelled improper. Included in the proper category are those solutions labelled weakly evolutionary by Jeffrey & Taniuti (1964). A proper shock is not necessarily physically realized because of the possibility that the evolutionary condition is too weak as a stability criterion. In order to provide the simplest possible form of these criteria, the existence of the following hierarchy among them is proved.

(d) *For properly formulated solutions to the de Hoffmann–Teller relations, the partial evolutionary condition* (c, i) *implies the entropy condition* (b); *the entropy condition implies the strong reality condition* (a, ii); *and the strong reality condition implies the weak reality condition* (a, i); *none of the converses is true.*

The proof is essentially to show that the intervals of ξ in (3.3), over which the various conditions are satisfied, are nested. Thus, the theorem is suggested and proved from the convenient form of (3.3), which has the following properties. First, the function in the denominator of (3.3) is

$$d(\xi, \cos^2\beta, \gamma) \equiv (\xi-1)^2 - \xi[\xi-\gamma/(\gamma-1)]\cos^2\beta \geqslant 0 \tag{4.1}$$

and $d = 0$ only for the switch-on shocks. This is seen by solving $d = 0$, which for $\xi \neq 1$ and $1 < \gamma < 2$ gives

$$\cos^2\beta = \frac{(\xi-1)^2}{\xi[\xi-\gamma/(\gamma-1)]},$$

or, equivalently,

$$\sin^2\beta = \frac{[(2-\gamma)/(\gamma-1)]\,\xi+1}{\xi[\xi-\gamma/(\gamma-1)]},$$

the first of which is negative for $\xi < \gamma/(\gamma-1)$, and the second of which is negative for $\xi > \gamma/(\gamma-1)$, so long as $\xi \neq 1$. When $\xi = \gamma/(\gamma-1)$, $d = (\gamma-1)^{-2} > 0$, so d is positive and non-zero for $\xi \neq 1$. For $\xi = 1$, $d = \cos^2\beta/(\gamma-1)$, which is positive, except for $\cos^2\beta = 0$, the switch-on shock, completing the proof.

$$\left.\begin{array}{l}\text{The numerator of (3.3) is positive for } \xi_{s0} < \xi < 1 \text{ or } \xi_{f0} < \xi\text{; it}\\ \text{vanishes only when } v_{0n} \text{ is one of the upstream characteristic}\\ \text{velocities } c_{s0},\, b_{0n} \text{ or } c_{f0}\text{, or if the downstream flow is Alfvénic}\\ (\xi = 1).\end{array}\right\} \quad (4.2)$$

This follows from the fact that the function

$$n(\xi, \cos^2\beta, \gamma) \equiv \xi^2\sin^2\beta - (1+s_0)\,\xi + s_0$$

vanishes only when ξ equals

$$\left.\begin{array}{ll} & \xi_{s0} = \{1+s_0-[(1+s_0)^2-4s_0\sin^2\beta]^{\frac{1}{2}}\}/(2\sin^2\beta) = c_{s0}^{+}/b_{0n}^{+2} = c_{s0}^2/\sin^2\beta,\\ \text{or} & \xi_{f0} = \{1+s_0+[(1+s_0)^2-4s_0\sin^2\beta]^{\frac{1}{2}}\}/(2\sin^2\beta) = c_{f0}^{+2}/b_{0n}^{+2} = c_{f0}^2/\sin^2\beta,\end{array}\right\} \quad (4.2a)$$

where (Herlofson 1950; van de Hulst 1951) $c_{s0}^2 \leqslant b_{0n}^2 \leqslant c_{f0}^2$ is always true.

When (4.2) is satisfied, $\eta = 0$ and the shock is of zero strength; thus

$$\xi = (v_{0n}^{+}/b_{0n}^{+})^2$$

and v_{0n} must be an upstream characteristic speed. The remainder of (4.2) follows from the signs of $n(\xi, \cos^2\beta, \gamma)$ and $(\xi-1)$ away from these roots of $n = 0$.

$$\left.\begin{array}{l}\text{The condition } c_{s0} \leqslant v_{0n} \leqslant b_{0n} \text{ is satisfied if and only if}\\ \qquad\qquad \xi_{s0} \leqslant \xi \leqslant \xi_{soff},\\ \text{where } \xi_{s0ff} \text{ is the smaller root of}\\ \quad f_1(\gamma, s_0, \beta, \xi) \equiv (\gamma+1)\sin^2\beta(\xi-1)^2\\ \qquad\qquad -[\gamma+2s_0-(\gamma+2)\sin^2\beta]\,(\xi-1)-\cos^2\beta = 0\\ \text{and where the upper and lower equalities correspond.}\end{array}\right\} \quad (4.3)$$

The first step here is to note that $v_{0n}^2 - c_{s0}^2$ or c_{f0}^2 can have three real roots in some cases, as may be seen from the Germain–Shercliff studies. One root is always the zero-strength shock discussed above, and the other two roots, when real, are always trans-Alfvénic shocks. For the c_{s0} case, this means that $\xi > 1$ is outside the ξ range of (4.3), and v_{0n} is either greater or less than c_{s0} in the ξ range of (4.3). By proving that $v_{0n} = b_{0n} > c_{s0}$ at $\xi = \xi_{soff}$, we prove also that v_{0n} is greater than c_{s0}.

Since $v_{0n} = b_{0n}$ is identical to $A_1^2 = 1$, and $A_1^2 = \bar{\rho}\xi$ from (2.11) and the definition of ξ, it follows that

$$A_1^2 - 1 = \bar{\rho}\xi - 1 = \eta\xi + \xi - 1 = \frac{(\xi-1)}{\gamma-1}\frac{2n+(\gamma-1)\,d}{d} = \frac{(\xi-1)}{\gamma-1}\frac{f_1}{d}.$$

Thus, except for $\xi = 1$, $A_1^2 = 1$ where f_1 vanishes. Because the first and third coefficients in f_1 are always of opposite signs, the roots, designated ξ_{off} and $\xi_{A(3)}$, must always be real and, respectively, less and greater than unity. Also, since $f_1 = 2n+(\gamma-1)\,d$, with d always positive, the first root of $f_1 = 0$ must always occur after n turns negative. That is, $\xi_{soff} > \xi_{s0}$ and the second root must satisfy $\xi_{A(3)} < \xi_{f0}$ for the same reason. Thus, we have $\xi_{s0} < \xi_{soff} < 1$ and $1 < \xi_{A(3)} < \xi_{f0}$ for all cases. Relation (4.3) follows from the first of these and the knowledge that $v_{0n} = c_{s0}$ only once for $\xi < 1$. The subscripts *soff* and $A(3)$ are used because these ξ values correspond respectively to the switch-off shock and the third solution for $A_1 = 1$, the first $A_1 = 1$ solution being the Alfvén shock.

$$\text{The condition } v_{0n} \geqslant c_{f0} \text{ holds if and only if } \xi \geqslant \xi_{f0}. \tag{4.4}$$

This follows easily from the fact that $v_{0n} = c_{f0}$ has only one root for $\xi > 1$, at $\xi = \xi_{f0}$, and the fact that, at $\xi = \xi_{A(3)}$, $v_{0n} = b_{0n} < c_{f0}$. Therefore, $v_{0n} - c_{f0}$ must be positive above ξ_{f0}, since it is negative below, and has a single zero at ξ_{f0}.

Now, (4.1) and (4.2) together show that the entropy condition (b) is identical to $\xi_{s0} \leqslant \xi \leqslant 1$ or $\xi \geqslant \xi_{f0}$, while (4.3) and (4.4) show the evolutionary condition (c, i) to be identical to $\xi_{s0} \leqslant \xi \leqslant \xi_{soff} < 1$ or $\xi \geqslant \xi_{f0}$. Thus, (c, i) implies (b), but the converse is not true. A corollary is that *entropy and evolutionary conditions are identical for fast shocks.*

The remainder of theorem (d) is proved by similar methods, so only an outline need be presented. First, it can be shown that $\bar{\rho} = 0$ is a cubic in ξ, with the always-real root ξ_{R1} satisfying $0 < \xi_{R1} < \xi_{s0}$, while the other two roots ξ_{R2} and ξ_{R3}, when real, fall in the interval $[1, \xi_{f0}]$. Then it is shown that $\bar{p} = 0$ is a quintic in ξ, with its always-real root lying in $[\xi_{R1}, \xi_{s0}]$. It has no root in $(\xi_{s0}, 1)$, but, for $\xi \geqslant 1$, may have none or two in $[1, \xi_{R2}]$ and $[\xi_{R3}, \xi_{f0}]$, respectively. These latter results follow from the fact that $\bar{p} = 1$ only if $\bar{\rho} = 1$, while $\bar{p} < 0$ where $\bar{\rho} = 0$. Combination of these statements completes the proof of (d).

Now, Ericson & Bazer (1960) have proved that (c, i) and (b) together imply (c, ii). That is, slow shocks are subslow and fast shocks are subfast downstream if they are compressive and super-slow but subAlfvénic and super-fast upstream, respectively. Because (d) has shown that (c, i) implies (b), this result is strengthened to read (c, i) implies (c, ii). A very useful new result follows.

(e) *The partial evolutionary condition* (c, i) *applied to a properly formulated problem is necessary and sufficient to guarantee that a solution to the shock relations* (2.1)–(2.9) *exists, and is real-valued, compressive and evolutionary.*

Thus, theorem (e) reduces the reality, entropy and evolutionary conditions to a single criterion, which, however, still contains upstream and downstream restrictions, since fast and slow are not simply adjectives, but restrictions on ξ as used here. This, along with (d), represents a unification and extension of the theorems on the relation between entropy and evolutionary conditions presented by Ericson & Bazer (1960), Taniuti (1962) and Lynn (1971); the latter requires a slight rewording as noted in Bertram (1973). Because of its necessary and sufficient form, (*e*) is clearly the strongest such theorem which can be proved.

5. Proper solutions

Beyond clarification of the roles of conditions (*a*)–(*c*) provided by (*e*), we obtain from (*e*) the new ability of constructing an algorithm for directly computing the proper shock solutions, without first computing and eliminating the improper solutions. This can greatly simplify analysis of any flow in which magnetogasdynamic shocks are present.

Consideration of the particular cases is begun with the one-dimensional formulation, for which the discussion of § 4 showed that solution (3.3) is proper if and only if

$$\xi_{s0} \leqslant \xi \leqslant \xi_{soff} \quad \text{or} \quad \xi \geqslant \xi_{f0}. \tag{5.1}$$

This is completely equivalent to Bazer & Ericson's earlier results with h as the strength parameter. However, the single-valuedness of (3.3), its use of the classifying parameter ξ as the strength parameter and its suggestive form all render this formulation more convenient.

In the aligned-fields formulation, it has been shown that (c, i) is equivalent to the following bounds on the strength parameter $\bar{\rho}$ (Bertram 1973):

$$\left.\begin{aligned} 1 \leqslant \bar{\rho} \leqslant \min\{A_1^2, M_1^{*2}\} \quad &\text{for} \quad A_1 > 1, \quad M_1 = (A_1^2/s_0)^{\frac{1}{2}} > 1, \\ \max\{1, \min\{\bar{\rho}_\pi, M_1^{*2}\}\} \leqslant \bar{\rho} \leqslant \max\{\bar{\rho}_\pi, M_1^{*2}\} \quad &\text{for} \quad s_0/(1+s_0) < A_1^2 \leqslant 1, \end{aligned}\right\} \tag{5.2}$$

where

$$\bar{\rho}_\pi \equiv A_1^2 + \frac{M_1^{*2}}{2(\gamma+1)}\left\{(\gamma+1-\gamma A_1^2) + \left[(\gamma+1-\gamma A_1^2)^2 - 4\frac{\gamma+1}{M_1^{*2}}A_1^2(A_1^2-1)\right]^{\frac{1}{2}}\right\}, \tag{5.2a}$$

in which $\bar{\rho}_\pi$ is, when real, the largest of the three roots of (3.1) with $\cos^2\beta = 1$.

Since both (5.2) and the equivalent of (5.1) have appeared in the references listed, the proper-shocks-only computational ability is not new for these cases. However, since no such relations have appeared for the two-dimensional nonaligned case (see Lynn 1966; Morioka & Spreiter 1969), those presented below are novel.

Now, when the two-dimensional representation is used, (3.1) is to be solved for $\bar{\rho}$ with β and A_1 obtained from (3.4). From this procedure, three values of $\bar{\rho}$ may be obtained. However, from (5.2) and the fact that $\bar{\rho}(\beta)$ is a monotone function in the ranges of (5.2), it follows that there is one and only one proper solution for each β which satisfies (c, i). Thus, constructing the equivalent of (5.1) and (5.2) for this case must consist of two parts: first, writing (c, i) in the form natural to this case, in terms of β and the characteristic angles ω of the steady flow; second, laying down a method of solution to the cubic (3.1) which always converges to the proper $\bar{\rho}$ root.

Even this first step cannot be carried out analytically, since the magneto-acoustic characteristics satisfy a quartic equation for $\tan(\omega+\psi_0)$, written here as equation (5.5), which has not been solved in closed form. Only the Alfvén

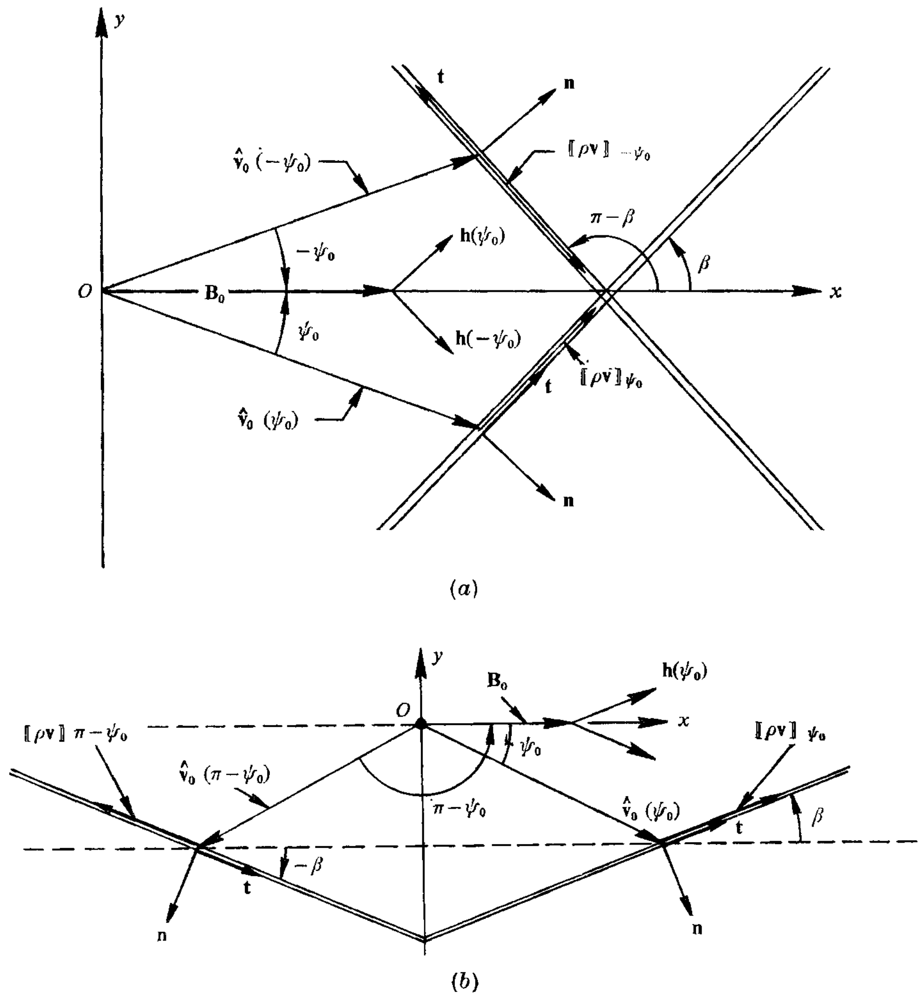
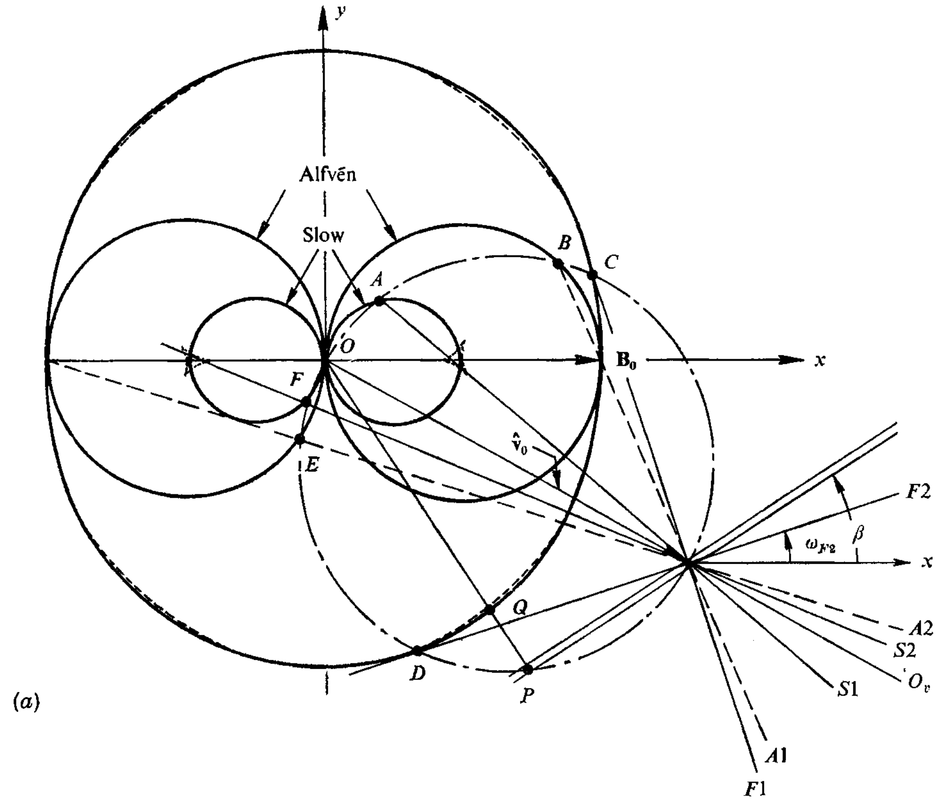


FIGURE 2. Symmetries of the two-dimensional solutions. (*a*) The transformation $\psi_0 \to -\psi_0$ with $\beta \to \pi-\beta$. (*b*) The transformation $\psi_0 \to \pi-\psi_0$ with $\beta \to -\beta$. Shock with ψ_0 is aligned; shock with $\pi-\psi_0$ is anti-aligned.

characteristic angles can be explicitly written down, as in (5.5*a*). This difficulty is largely overcome by use of the graphical construction (McCune & Resler 1960; Sears 1960) based on the Friedrichs diagram (Friedrichs & Kranzer 1958), which allows general conclusions to be drawn by enumerating the possible characteristic configurations.

In order to minimize the number of cases considered, the following symmetry properties of the solution are exploited. First, for a fixed upstream state (constant γ, s_0, $\mathbf{B}_0$, A_0 and ψ_0), all possible shocks are obtained when β ranges over $0 \leqslant \beta \leqslant \pi$. This follows since the substitution $\beta \to \beta+\pi$ simply rotates all of figure 1 by π, and therefore represents the same shock.

Second, all possible cases for fixed γ, s_0, $\mathbf{B}_0$ and A_0 are covered for $0 \leqslant \psi_0 \leqslant \pi$, because the substitution $\psi_0 \to -\psi_0$ with $\beta \to \pi-\beta$ gives the same A_1, $\cos^2\beta$ and



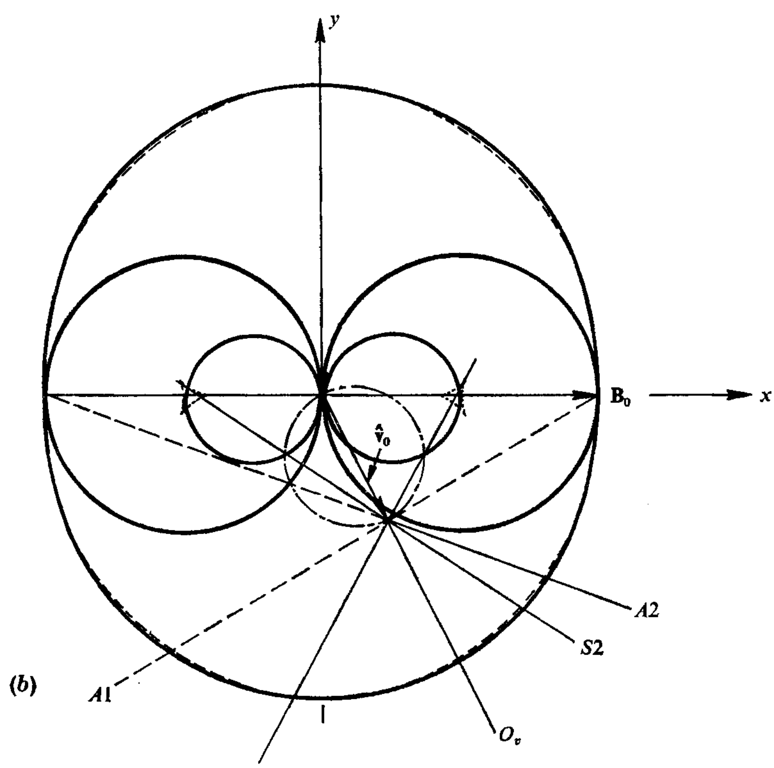
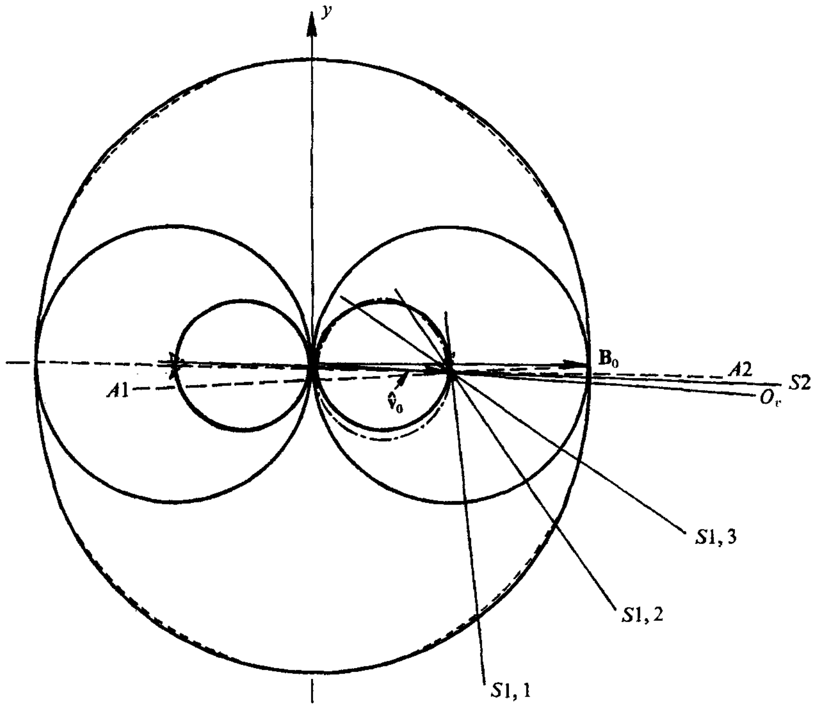


FIGURE 3 (a, b). For legend see p. 328.



(c)

FIGURE 3. Upstream evolutionary condition for two-dimensional case. — – —, v_{0n} circle; ——, characteristic velocity locus; – – –, Friedrichs diagram (or Alfvén characteristic); ═, shock surface. (*a*) $s_0 = \frac{1}{4}$, $\psi_0 = 30°$, $A_0 = 1{\cdot}5$. Two fast, two Alfvén and two slow characteristics. Evolutionary between $F1$ and $F2$ ($\omega_{F2} - \pi \leqslant \beta \leqslant \omega_{F1}$) or between slow and Alfvén ($\omega_{A1} \leqslant \beta \leqslant \omega_{S1}$, $\omega_{S2} \leqslant \beta \leqslant \omega_{A2}$). (*b*) $s_0 = \frac{1}{4}$, $\psi_0 = 60°$, $A_0 = 0{\cdot}5$. Two slow and two Alfvén characteristics. Evolutionary between slow and Alfvén ($\omega_{A1} \leqslant \beta \leqslant \omega_{S1}$, $\omega_{S2} \leqslant \beta \leqslant \omega_{A2}$), (*c*) $s_0 = \frac{1}{4}$, $\psi_0 = 3{\cdot}46565°$, $A_0 = 0{\cdot}50304$. Four slow and two Alfvén characteristics. Evolutionary between slow and Alfvén ($\omega_{A1} \leqslant \beta \leqslant \omega_{S1,1}$, $\omega_{S2} \leqslant \beta \leqslant \omega_{A2}$) and between $S1, 2$ and $S1, 3$ ($\omega_{S1,2} \leqslant \beta \leqslant \omega_{S1,3}$). See figure 4.

hence η, ξ, etc. values, with only $\mathbf{h} \rightarrow -\mathbf{h}$ and $[\![u_t]\!] \rightarrow -[\![u_t]\!]$ being changed. Thus, these two shocks are simply mutual reflexions in the x axis ($\mathbf{B}_0$ direction) and are physically identical (figure 2*a*).

Finally, $0 \leqslant \psi_0 \leqslant \frac{1}{2}\pi$ is sufficient for calculational purposes. To see this, consider the two shocks related by $\psi_0 \rightarrow \pi - \psi_0$ and $\beta \rightarrow -\beta$. Then $A_1 \rightarrow -A_1$, $[\![u_t]\!] \rightarrow -[\![u_t]\!]$ and all other variables are unaffected. As may be seen in figure 2 (*b*), these two shocks are related by reflexion of the induction vectors in the x axis, while the velocity vectors are reflected through the y axis. This means that the two shocks are physically distinct. However, they are so similar that they need not be considered separately. The essential difference is that one has $\hat{\mathbf{q}}_0$ aligned ($A_1 > 0$) and the other has $\hat{\mathbf{q}}_0$ anti-aligned ($A_1 < 0$) with $\mathbf{B}_0$. Related symmetries about the velocity vector have been discussed by Lynn (1966).

Now, to examine possible characteristic configurations for $0 \leqslant \psi_0 \leqslant \frac{1}{2}\pi$ conveniently, the Friedrichs diagram construction is employed in the following slightly modified manner. Rather than plotting $-\hat{\mathbf{v}}_0$ on the diagram and interpreting the line segment from the tip of $-\hat{\mathbf{v}}_0$ to the point of tangency as the characteristic surface, $\hat{\mathbf{v}}_0$ is plotted and the characteristic angles ω are measured from the x direction to the tangent extended past the tip of $\hat{\mathbf{v}}_0$. The resulting characteristics occur in pairs of fast (F), slow (S) and Alfvén (A) characteristics, and can have only the combinations shown in figure 3 for $0 \leqslant \psi_0 \leqslant \frac{1}{2}\pi$ (Bazer & Fleishman 1959; Weitzner 1961; Lynn 1962). The member of the pair whose tangent point is farthest to the right on the Friedrichs diagram is distinguished by a 1, making $F1$, $S1$ and $A1$. The remaining member is denoted by $F2$, $S2$ or $A2$. When there are four slow characteristics as in figure 3(c), three of them come from the same portion of the characteristic locus, and are further distinguished by adding a subscript 1, 2 or 3 so that $0 \leqslant \omega_{S1,1} < \omega_{S1,2} < \omega_{S1,3} \leqslant \pi$. Note that this arrangement always has $F2$, $S2$ and $A2$ above O_v, while $F1$, $A1$ and at least one $S1$ lie below O_v, where O_v denotes $\hat{\mathbf{v}}_0$ extended, that is, the streamline characteristic.

Once the characteristics have been obtained, a shock can be indicated by drawing the shock plane through the tip of $\hat{\mathbf{v}}_0$ at an angle β to $\mathbf{B}_0$. The relation of v_{0n} to the characteristic velocity then can be simply determined by drawing the circle with $\hat{\mathbf{v}}_0$ as diameter. This is possible since a ray OP from the origin to the point P on the circle where the shock intersects it is necessarily normal to the shock surface, so $|OP| = v_{0n}$ (see figure 3a). It follows that $v_{0n} \lessgtr c_0$ is true whenever the point P is inside or outside the characteristic velocity locus, so the upstream part of (c, i) can be read directly off the v_{0n} circle and the characteristic velocity locus. In fact, we can dispense with the Friedrichs diagram since the intersections of the v_{0n} circle with the characteristic velocity locus (points O, A, B, C, D and E going clockwise around the circle in figure 3a) have $v_{0n} = c_0$, and therefore provide an alternative method of constructing characteristics. However, the Friedrichs diagram construction will be used both here and in part 2 because it is familiar and somewhat easier to work with.

It follows from the construction described above that $c_{s0} \leqslant v_{0n} \leqslant b_{0n}$ holds for all β values for which the corresponding segment of the v_{0n} circle falls between the slow and Alfvén characteristic velocity loci, such as arcs AB and EF in figure 3(a); $v_{0n} \geqslant c_{f0}$ for those β for which the v_{0n} circle is outside the fast characteristic velocity locus, such as CD in figure 3(a). While it is easy to see in figure 3(b) that only the regions between $S1$ and $A1$, and between $S2$ and $A2$ satisfy the evolutionary condition upstream, it is necessary to consider the slow locus in detail to confirm the conclusion stated for figure 3(c). Thus, figure 4 shows a slow characteristic velocity locus with its x-length/y-length ratio exaggerated in order to enhance clarity. Since the slow characteristic velocity locus is always elongated in the x direction, the figure is qualitatively correct for all cases. The caption statement is evident for the $\hat{\mathbf{v}}_0$ shown, and the reader may confirm that it is true also when the tip of $\hat{\mathbf{v}}_0$ lies outside the characteristic velocity locus, but still within the Friedrichs diagram triangle, such as at P in figure 4. The condition $v_{0n} \leqslant b_{0n}$ must hold for every point on the v_{0n} circle between O and the $S1,1$ intersection with the slow characteristic velocity locus, so that this condition

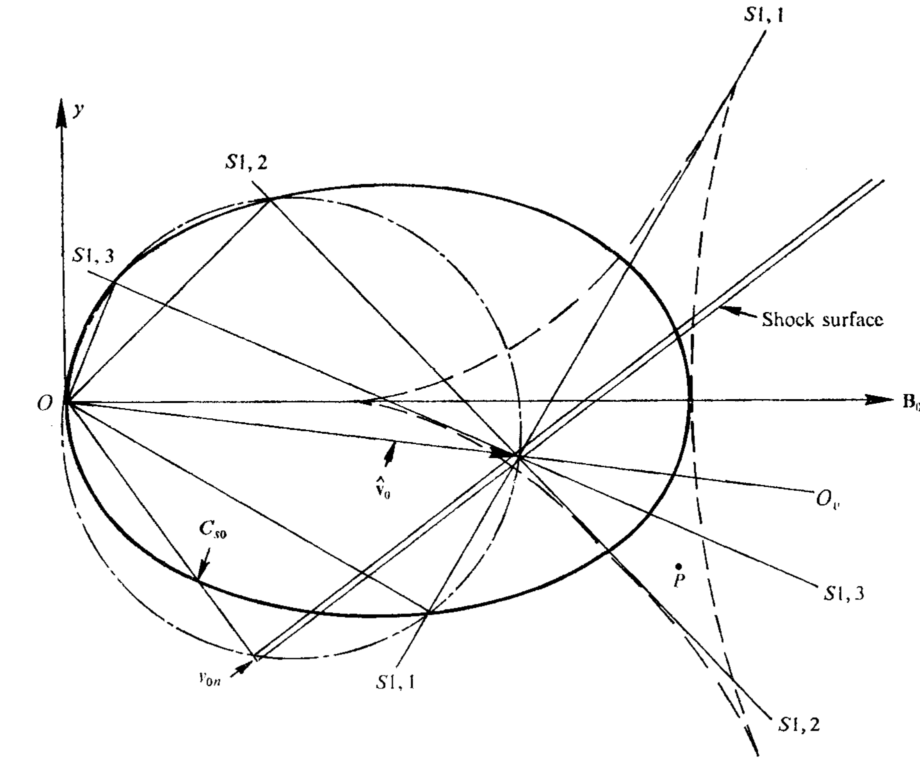


FIGURE 4. Proof that $v_{0n} \geqslant c_{s0}$ always requires $0 \leqslant \beta \leqslant \omega_{S1,1}$ or $\omega_{S1,2} \leqslant \beta \leqslant \omega_{S1,3}$ to be true when there are four slow characteristics. — – —, v_{0n} circle; ——, slow characteristic velocity locus; – – – (slow) Friedrichs diagram.

excludes only a portion between O and $S1,1$, and the conclusion of figure 3(c) is proved.

$$\left.\begin{array}{l}\text{All the above results may be summarized by the following simple statement. \textit{Whether there are four or six characteristics,}}\\ \textit{the upstream evolutionary condition } c_{s0} \leqslant v_{0n} \leqslant b_{0n} \textit{ or } v_{0n} \geqslant c_{f0}\\ \textit{is satisfied between every second pair, starting with the } S2\text{–}A2 \textit{ pair}\\ \textit{for } 0 \leqslant \psi_0 \leqslant \tfrac{1}{2}\pi.\end{array}\right\} \quad (5.3)$$

The geometric form of this is shown in figure 5. It might be remarked that the convexity of the v_{0n} circle and the characteristic velocity loci are sufficient to prove statement (5.3) directly, since portions of the v_{0n} circle inside and outside the velocity loci must necessarily alternate between intersections.

Now it remains only to compute the proper η from (3.1), which may be written as

$$P(\eta) = \eta^3 + C_2\eta^2 + C_1\eta + C_0 = 0, \tag{5.4}$$

where

$$C_2 = \gamma\frac{A_1^2 M_1^{*2}}{\gamma+1}\cos^2\beta - 2(A_1^2-1) - (M_1^{*2}-1),$$

$$C_1 = (A_1^2-1)^2 + 2(A_1^2-1)(M_1^{*2}-1) - [1+(\gamma-1)(A_1^2-1)]\frac{A_1^2 M_1^{*2}}{\gamma+1}\cos^2\beta$$

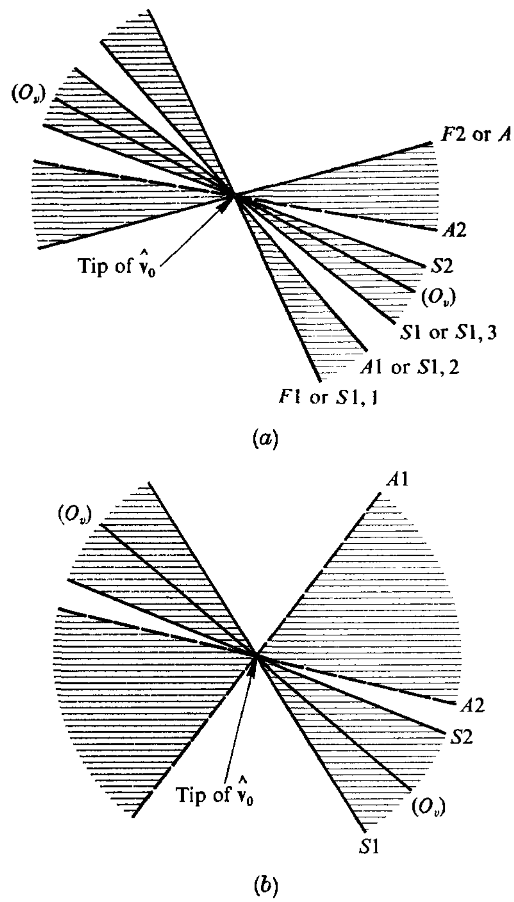


FIGURE 5. Hatched regions violate evolutionary condition (*c*, i); unhatched regions always have single proper shock at each β. (*a*) Six real characteristics. (*b*) Four real characteristics. $0 \leqslant \psi_0 \leqslant \frac{1}{2}\pi$.

and

$$C_0 = -(A_1^2-1)\left[(A_1^2-1)(M_1^{*2}-1)-\frac{2}{\gamma+1}A_1^2M_1^{*2}\cos^2\beta\right].$$

Since this is to be solved numerically, we seek only the analytical properties of (5.4) which guarantee efficient numerical solution. Because of its fast convergence and simplicity, the Newton–Raphson method is used here, so we need only supply a starting value η_s which is close enough to the actual root η, that is, such that there is neither a critical nor inflexion point of $P(\eta)$ between η_s and η (see Henrici 1964). For the fast-shock case, the Germain–Shercliff analysis in figure 6 indicates that all three roots of (5.4) must be compressive [note that (5.3) has already selected point 1 in figure 6 as the upstream state], and that the proper root is the smallest of the three. It follows that $P(\eta)$, being cubic, must have both its critical points and its only inflexion point above the proper η, so that $\eta_s = 0$ guarantees convergence to η. For the slow-shock case, the evolutionary condition (5.6) has selected point 3 as the upstream state, so that (5.4)

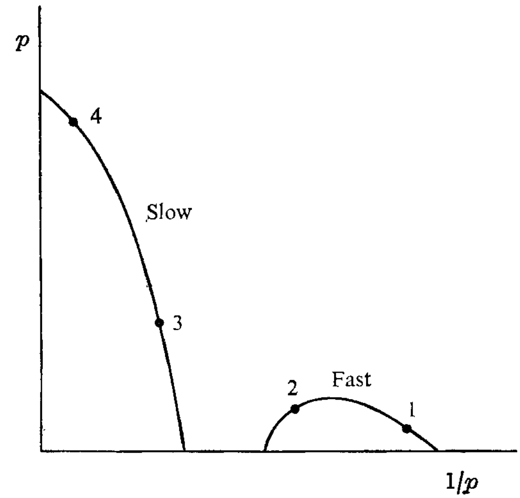


FIGURE 6. Rayleigh line (after Shercliff). Numbered points satisfy shock relations with entropy increasing 1 to 4.

can have only one positive root, and that is the proper η. Thus, taking $\eta_s = 2/(\gamma-1)$ guarantees convergence to η from above, since the critical points and inflexion point of $P(\eta)$ must lie to the left of the proper η, which must be smaller than η_s according to the reality condition (a, i).

In summary, then, the entire algorithm, equivalent to equations (5.1) and (5.2), for computing only proper two-dimensional shocks, consists of the following. Magneto-acoustic characteristic angles are obtained from the roots of

$$[(A_0^2-1)(M_0^2-1)\sec^2\psi_0-\tan^2\psi_0]\tau^4-2\tan\psi_0\tau^3-(A_0^2+M_0^2-1)$$
$$\times\sec^2\psi_0\tau^2-2\tan\psi_0\tau+\tan^2\psi_0 = 0, \quad (5.5)$$

where $\tau \equiv \tan(\omega+\psi_0)$ and $M_0^2 \equiv A_0^2/s_0$. Equation (5.5) is a slightly rearranged version of Lynn's (1966) equation 21. Alfvén characteristic angles are found from

$$\tan\omega_{Ai} = A_0\sin\psi_0/(-A_0\cos\psi_0\pm 1), \quad (5.5a)$$

with $A1$ and $A2$ corresponding, respectively, to the plus and minus signs in (5.5*a*).

Proper β ranges may be identified once (5.5) has been solved by noting that the $S2$ characteristic is always the first above O_v, so that its τ root is always the smallest positive root of (5.5). It follows that (5.3) can be given the general analytical form

$$\omega_1 \leqslant \beta \leqslant \omega_2, \quad \omega_3 \leqslant \beta \leqslant \omega_4, \quad \omega_5 \leqslant \beta \leqslant \omega_6 \quad \text{(when } \omega_5 \text{ and } \omega_6 \text{ are real)}, \quad (5.6)$$

where $\omega_1 \equiv \omega_{S2}$ = smallest positive τ-root of (5.5) and $\omega_i \leqslant \omega_{i+1}$ are characteristic angles, both Alfvén and magneto-acoustic, ordered from ω_{S2} to $\pi-\omega_{S2}$ in ascending order. Thus, the second step in the algorithm is to test whether the β value prescribed falls into the ranges of (5.6). If it does not, there are no proper shock solutions and calculation stops.

When (5.6) has been satisfied, the A_1 value is obtained from (3.4), and M_1^{*2}

is computed from A_1, so that all coefficients in (5.4) are determined. Then (5.4) can be solved by a Newton–Raphson method, with

$$\eta_s = \frac{1-\operatorname{sgn}(A_1^2-1)}{\gamma-1}. \tag{5.7}$$

The complete two-dimensional algorithm, then, consists of solving (5.4) with a starting value of (5.7), if test (5.6), with values from (5.5), is passed.

The crossed-fields solution, equation (3.5*b*), when subject to equation (3.5*a*) is properly formulated, and the proper $\bar{\rho}$ value will thus be given by (3.5*b*) when ξ is chosen so that β falls in the ranges of (5.6), according to theorem (*e*). Thus, we must have

$$\xi_{s0} \leqslant \xi \leqslant \xi_{soff} \quad \text{or} \quad \xi_{f0} \leqslant \xi, \tag{5.8}$$

where ξ_{f0} and ξ_{s0} are the roots of the crossed-fields characteristic equation, since $\bar{\rho} = 1$ on the characteristics, implying that $\xi = A_0^2 \cot^2 \omega$. Thus they satisfy

$$(A_0^2 - 1 - s_0)\,\xi^2 - (A_0^2 + s_0 A_0^2 - s_0)\,\xi + s_0 A_0^2 = 0, \tag{5.8a}$$

which always has a positive real root $\xi_{s0} < 1$, and has a positive root $\xi_{f0} > 1$ whenever $A_0^2 \geqslant 1 + s_0$. The value ξ_{soff} is found most easily by setting $A_1^2 = 1$ in (3.5) and factorizing $\xi - 1$ to obtain

$$(\gamma+1)\,A_0^2(\xi-1)^2 + [\gamma + 2s_0 + 2(s_0-1)\,A_0^2]\,(\xi-1) - 1 = 0, \tag{5.8b}$$

the smaller root of which is ξ_{soff} and the larger is $\xi_{A(3)}$.

In closing, we note that necessary and sufficient conditions for each of the limit and singular solutions to be proper have been given in the discussion of these cases in § 3, so that the above rules cover all cases.

6. Classification and discussion of solutions

Construction of a general classification scheme is motivated by the desire to gain an overall understanding of the possible shock properties. The task is begun here by distinguishing between an individual shock solution, for which every parameter has a specific numerical value, and a shock polar, the set of individual solutions obtained by varying the strength parameter over all possible values, while holding the upstream parameters fixed. A class of individual solutions can then be defined by some shared property, while properties possessed by a shock polar should be used to classify polars.

There have been essentially two types of classification schemes based on individual shock properties. The first is the set of special-case classes, such as normal shocks ($\beta = \frac{1}{2}\pi$) and perpendicular shocks (de Hoffmann & Teller 1950), etc. The second is the set of general classes of fast, slow and intermediate introduced by Friedrichs & Kranzer (1958), but made exhaustive by the definitions of Bazer & Ericson (1959), who noted that inclusion of the limit shocks in the class (e.g. perpendicular shocks classed as fast, etc.) made it possible to include every shock in one of the three classes. These individual shock classes are summarized in table 1, from which improper solutions are excluded on the grounds that it serves no purpose to classify them, since the rules presented in § 5 allow

All proper shock solutions					
Fast ($\xi > 1$)		Intermediate ($\xi = 1$)		Slow ($\xi < 1$)	
Perpendicular shocks ($\xi \to \infty$, $\beta \to 0$)	Simple fast (finite ξ)	Switch-on ($\beta = \frac{1}{2}\pi$)	Alfvén ($A_1 = 1$)	Switch-off ($A_1 = 1$)	Simple slow ($A_1 < 1$)

TABLE 1. Classification of individual shock solutions. In addition, there are many special-case classes; e.g., normal shocks, zero-strength shocks, etc.

us to avoid them. Also, since the intermediate class contains only two subclasses, it is redundant, but is retained for completeness.

A detailed study of shock polars and their classification is contained in part 2, but we make the following observations, which cast some light on the individual solution classification. A shock polar is defined by holding some set of upstream parameters constant and varying a strength parameter. Therefore, there is a one-to-one correspondence between general polar types, distinguished by the set of parameters held constant, and the representations discussed in § 3. Within the one-dimensional shock polar type, some individual polars will have the parameter h as a monotone function of the strength parameter; others will have h reaching a maximum and then decreasing, so that each h within a certain range will correspond to two shocks of the polar. These polars were labelled type 1 *shock* and type 2 *shock*, respectively by Bazer & Ericson (1959). Clearly, since the defining property is a property of the whole polar instead of an individual solution, it is more logical to attach the class adjective to the polar; thus, type 1 polar, etc. A second example of the same ambiguity is the terminology 'incomplete fast gas shock completed by the incomplete switch-on shock', which is applied to the one-dimensional unsteady polar with $\beta = \frac{1}{2}\pi$, that is, the normal-shock polar which is made up of gas shock and switch-on shock branches (Bazer & Ericson 1959). It is the above reasoning which leads to the proposal that classification of individual shock solutions be considered complete with the scheme of table 1, plus the special-case classes, and that a second scheme be derived from the properties of, and applied to, the polars as is done in part 2.

Now, as stated in the introduction, this study has had two major goals: increasing insight into the possible properties of magnetogasdynamic shocks, and arranging formal solutions for efficient numerical calculations in applications. The first goal has so far been approached only to the most preliminary extent. In fact, it can only be fully achieved by considering the shock polars. Since each of these contains all the possible downstream states which can be reached from a single upstream state, as viewed from a particular co-ordinate system, mastery of the properties of each polar represents an understanding of a whole continuum of individual shock solutions. Accordingly, part 2 of this study pursues the construction and classification of the polars, for which purpose the forms of the solutions in § 3 were mainly developed.

The second goal has been achieved to an extent, perhaps greater than is obvious. First, the rules of § 5, applied directly to the strength parameter in each solution, render all these solutions single-valued functions of the strength parameter. This implies that any application in which the strength parameter is dictated as a boundary value is trivial, while a simple iteration can be set up if the dictated boundary value is not the strength parameter. To consider a specific application, both the studies of Lynn (1966) and of Morioka & Spreiter (1969) were motivated by a wish to employ the deflexion angle of the velocity vector as a strength parameter in a non-aligned two-dimensional flow, because the magnetoaerodynamic boundary-value problem, in which the body shape dictates the flow direction, is of central interest. The result of Lynn's study was an equation cubic in the tangent of the deflexion angle, and of degree eight in the tangent of the shock angle. Clearly, the simpler procedure is to solve it for the deflexion angle with the shock angle as strength parameter. In the Morioka & Spreiter formulation, a combination of shock and deflexion angle is the unknown, which satisfies a cubic equation with coefficients given as functions of shock angle as strength parameter. The only advantage possessed by either form, then, over the algorithm presented here is the elimination of a single computational step, that of computing the deflexion angle after solving (5.4) for $\bar{p}$. In all three forms, it is still necessary to iterate on β to obtain the prescribed deflexion angle. The price paid for elimination of the single step is the necessity of computing all three roots of the cubic, and then eliminating two of them by some rule derived from reality, entropy and evolutionary conditions. Of course, application of (5.6) to Lynn's or Morioka & Spreiter's equations, along with the development of an algorithm equivalent to (5.7) to guarantee convergence to the proper root, could eliminate even the single extra step. The conclusion to be drawn from this is that the single-valuedness of the present solution is a computational advantage great enough to make this solution generally useful, no matter what boundary data may be supplied by the problem to which it is applied.

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